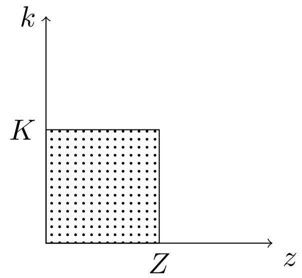
Physics with neutrons 1

Sebastian Mühlbauer, sebastian.muehlbauer@frm2.tum.de
Winter semester 2015/16
Exercise sheet 3
Due 2015–Nov–06

Lukas Karge, lukas.karge@frm2.tum.de, Tel.: 089-289-11774

Exercise 3.1

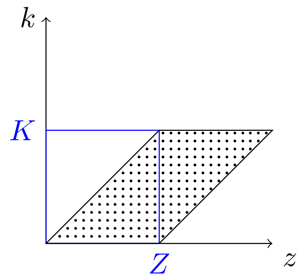
Consider a one-dimensional movement (along the z-axis) of $N \gg 1$ neutrons. The state of a single neutron is defined by its position in space z and wave vector k (we neglect the neutron spin). This state can be represented as a point in phase space. Let the density of these points be constant at time $t = 0$ in the area $0 \leq z \leq Z$ and $0 \leq k \leq K$ and zero elsewhere.
Calculate and sketch the phase space developing in time (i) for a force-free movement and (ii) in the gravitational field $\mathbf{g} = g\mathbf{e_z}$. Explain why the enclosed volume in phase space and the density of points is constant.



Solution. (i) In case of force-free movement we have

$$k_i(t) = k_i(0), \qquad z_i(t) = \frac{\hbar k_i(0)}{m_n} t + z_i(0)$$

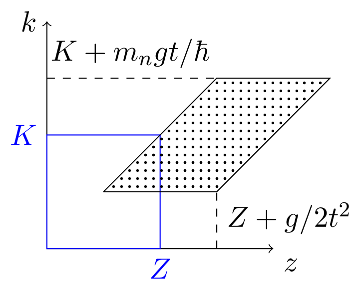
Every point in phase space translates along that trajectory. The rectangle is mapped to a parallelogram. Its volume stays constant and is true for any shape (we can partition complicated shapes into small rectangles) and that infers a constant *density* of phase space.



(ii) For the movement in gravitational field we get

$$k_i(t) = k_i(0) + \frac{m_n}{\hbar} g t, \qquad z_i(t) = \frac{g}{2} t^2 + \frac{\hbar k_i(0)}{m_n} t + z_i(0)$$

Again, we sketch the development in phase space.



Remark 3.1. This result can be generalized: We describe an ensemble of neutrons by a density in phase space:

$$\rho(r, p)\ d^3r\ d^3p.$$

Then Liouville's theorem states: The phase space volume occupied by particles is constant if only conservative forces [=derivable from a potential] act. □

EXERCISE 3.2

Look up the dispersion relation for magnons in ferromagnetic chains. Calculate the zone boundary energy of such a magnon with spin $S = 1$, a lattice constant $a = 2\,\text{Å}$ and different exchange constants $J = 1\,\text{meV}$ and $J = 30\,\text{meV}$. Draw the scattering diagrams for a neutron scattering experiment to measure the energy of magnons with a momentum transfer $Q = 0.05\,\text{Å}^{-1}$ within the first Brillouin zone using neutrons with $k_f = 2.57\,\text{Å}^{-1}$. Consider magnon creation and annihilation.
Finally, draw the dispersion curves for the two magnons within the kinematic plane given by

$$Q = \left[\frac{2m_n}{\hbar^2}\left(2E_i \mp \hbar\omega - 2\cos\Theta\sqrt{E_i \cdot (E_i \mp \hbar\omega)}\right)\right]^{1/2}.$$

Solution. The dispersion of a ferromagnetic magnon is given by

$$E(k) = 4JS(1 - \cos ka),$$

with the exchange energy J, the spin modulus S and the lattice constant a.
For small k, the cosine can be Taylor-expanded to yield

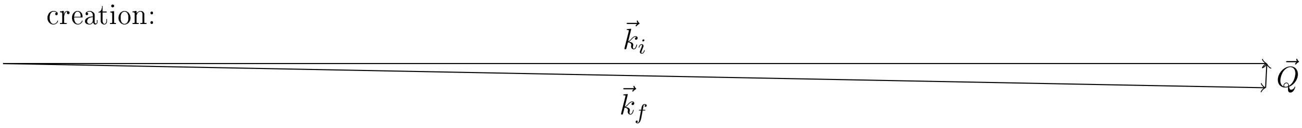
$$E(k) \approx 4JS\left(1 - 1 + \frac{k^2a^2}{2}\right) = 2JSa^2k^2 =: Dk^2,$$

with the "spin-wave stiffness" $D = 2JSa^2$. (In contrast, if one expands the dispersion of a phonon from ex. 2.1, it becomes $E(k) \approx Ak$, i.e. linear for small k.)
The parameters given in the exercise result in stiffness constants of $D_1 = 8\,\text{meV}\,\text{Å}^2$ and $D_2 = 240\,\text{meV}\,\text{Å}^2$, respectively. Now we want to do a scattering experiment for $Q = 0.05$Å again with $k_f = 2.57$Å.
From energy conservation we get, as for phonons,

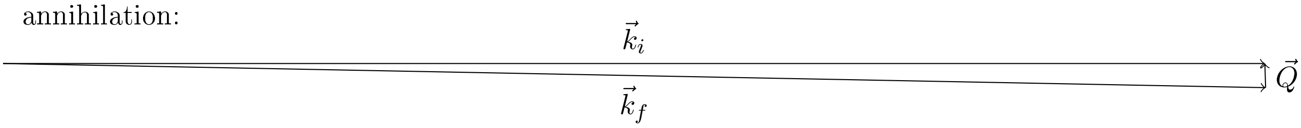
$$k_i = \sqrt{k_f^2 \pm \frac{2mE}{\hbar^2}} = \sqrt{k_f^2 \pm \frac{2mDQ^2}{\hbar^2}},$$

which results for $D = D_1$ in $k_i = 2.572$Å (magnon creation) and $k_i = 2.568$Å (magnon annihilation).
Scattering diagrams:

creation:

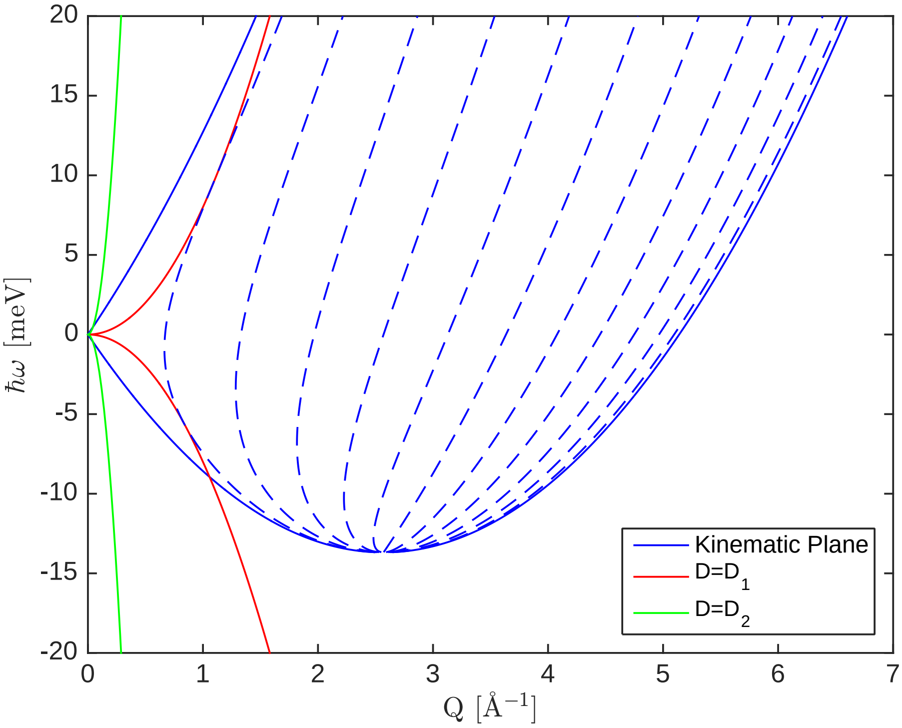


annihilation:



For $D = D_2$ the energies are much larger, and we get $k_i = 2.626$Å and $k_i = 2.513$Å, respectively, which cannot be reached at any scattering angle for this small q.
For fixed k_f, we can rewrite the scattering plane equation in terms of E_f:

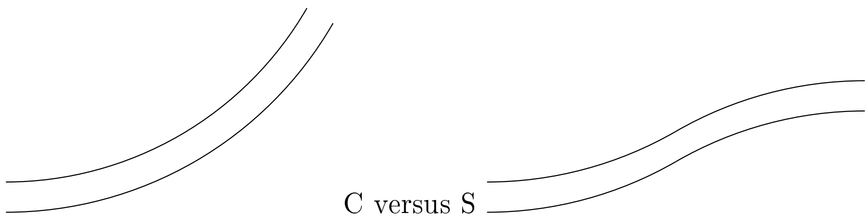
$$Q = \left[\frac{2m_n}{\hbar^2}\left(2E_f \pm \hbar\omega - 2\cos\Theta\sqrt{E_f \cdot (E_f \pm \hbar\omega)}\right)\right]^{1/2}$$



As you can see, the dispersion of the magnon with $D = D_2$ (green) lies almost completely out of the reachable scattering area. (The straight blue lines are the edges of the reachable area, for $\Theta = 0°$ and $\Theta = 180°$, while the dotted blue lines are drawn for $0° < \Theta < 180°$ in steps of 15°.) □

Exercise 3.3

1. To reduce the amount of γ radiation and fast neutrons that arrive at the instruments, many neutron guides are curved (C-shaped) so that no direct line of sight on the neutron source is possible. Modern neutron guides are usually S-shaped (SANS-1, TOFTOF,... at FRM-II).



C versus S

   What is the advantage of the S shape?

2. Suggest forms of neutron guides that
   a) focus a parallel beam onto a point-like sample
   b) focus a point-like source onto a point-like sample

   to increase the flux at small samples. What is the drawback of this focussing?

Solution. 1. There are actually two advantages:

a) The beam profile is symmetrical

b) In a C-shaped guide, fast neutrons may pass the curve by finding a path with many reflections (at small angles) at the outer side of the guide. In an S-shaped guide, these neutrons can do so only half way after which they run under a quite large angle into the wall and are not reflected any more.

2. a) A parabola

   b) An ellipse.

The drawback of this focussing is an increased divergence which cannot be used for example in high resolution diffraction experiments (Liouville's Theorem).

□